Chapter 0

A0.1 Prove the following vector product rules:
(a) $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$
(b) $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$
(c) $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$
(d) $\vec{A} \times (\vec{B} \times (\vec{C} \times \vec{D})) = \vec{B}(\vec{A} \cdot (\vec{C} \times \vec{D})) - (\vec{A} \cdot \vec{B})(\vec{C} \times \vec{D})$
(e) Show that the volume of a triclinic primitive unit cell ($a \neq b \neq c, \alpha \neq \beta \neq \gamma$) is:
$$V = \hat{e} \cdot (\hat{a} \times \hat{b}) = \hat{c} \cdot \hat{z} (ab \sin \gamma) = abc \sin \gamma$$
$$V = abc \sin \gamma \sqrt{1 - \cos^2 \beta - \left(\frac{\cos \alpha - \cos \beta \cos \gamma}{\sin^2 \gamma}\right)^2} = abc \sqrt{\sin^2 \beta \sin^2 \gamma - (\cos \alpha - \cos \beta \cos \gamma)^2}$$
(f) Show that the volume of a rhombohedral (trigonal) primitive unit cell ($a = b = c, \alpha = \beta = \gamma$) is:
$$V = \hat{c} \cdot (\hat{a} \times \hat{b}) = a^3 \sqrt{\sin^4 \alpha - \cos^2 \alpha (1 - \cos \alpha)^2} = a^3 (1 - \cos \alpha) \sqrt{1 + 2 \cos \alpha}$$

A0.2 Prove the following product rules:
(a) $\nabla \cdot (f \vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla f)$
(b) $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$
(c) $\nabla \times (f \vec{A}) = f(\nabla \times \vec{A}) - \vec{A} \times (\nabla f)$

A0.3 Prove the following second derivatives:
(a) $\nabla \cdot (\nabla \times \vec{A}) = 0$
(b) $\nabla \times (\nabla f) = 0$
(c) $\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$
(d) $\nabla^2 (fg) = g\nabla^2 f + 2\nabla f \cdot \nabla g + f\nabla^2 g$

A0.4 If $\vec{r}$ is the coordinate of a point with respect to some origin, with magnitude $r = |\vec{r}|$, $\hat{r} = \vec{r} / r$ is a unit radial vector, and $f(r)$ is a well-behaved function of $r$, show that
(a) $\nabla r = \hat{r}$, 
(b) $\nabla \frac{1}{r} = -\hat{r} \frac{1}{r^2}$, 
(c) $\nabla \cdot \hat{r} = 3$, 
(d) $\nabla \times \hat{r} = \frac{2}{r}$, 
(e) $\nabla \times \hat{r} = 0$, 
(f) $\nabla \times \frac{\hat{r}}{r^2} = 0$.
(g) \[ \nabla \cdot \left( \frac{\hat{r}}{r^2} \right) = 4\pi \delta^3 (\hat{r}), \]

(h) \[ \nabla^2 \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) = -4\pi \delta^3 (\vec{r} - \vec{r}'). \]

(i) \[ \nabla f(r) = \frac{df}{dr} \hat{r}, \]

(j) \[ \nabla \cdot (\vec{r} f(r)) = 2 f \hat{r} / r + \vec{r} \vec{f} \hat{r} / \vec{r}, \]

(k) \[ \nabla \times (\hat{r} f(r)) = 0, \]

(l) \[ (a \cdot \nabla) f(r) = \frac{f(r)}{r} [a - \hat{r}(a \cdot \hat{r})] + \hat{r}(a \cdot \hat{r}) \frac{\vec{f}}{\vec{r}}. \]

(m) \[ \nabla (\vec{r} \cdot a) = a + \vec{r} (\nabla \cdot a) + i (\vec{L} \times a), \] where \( \vec{L} = -i (\vec{r} \times \nabla) \) is the angular momentum operator.

(n) Given \( \tau = |\vec{r} - \vec{r}'| \) show that in spherical coordinates

\[ \tau = \sqrt{r^2 + r'^2 - 2 \vec{r} \vec{r}'} = \sqrt{r^2 + r'^2 - 2rr' \cos \psi} = \sqrt{r^2 + r'^2 - 2rr' \cos \theta' \cos \psi + \sin \theta \sin \theta' \cos (\phi - \phi')} \]

and \( \cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') \)

(o) Given \( \tau = |\vec{r} - \vec{r}'| \) show that in cylindrical coordinates

\[ \tau = |\vec{r} - \vec{r}'| = \sqrt{s^2 + s'^2 - 2ss' \cos (\phi - \phi')} + (z - z')^2 \]

(p) Prove: \[ \nabla \left( \frac{\vec{p} \cdot \hat{r}}{r^2} \right) = \frac{1}{r^3} [\vec{p} - 3 (\vec{p} \cdot \hat{r}) \hat{r}], \] where \( \vec{p} = p_x \hat{r} + p_\theta \hat{\theta} + p_\phi \hat{\phi} \) is a constant vector, and \( \vec{r} \neq 0 \).

(q) Prove: \[ \nabla \times \left( \frac{\vec{m} \times \hat{r}}{r^2} \right) = \frac{1}{r^3} [3(\vec{m} \cdot \hat{r}) \hat{r} - \vec{m}], \] where \( \vec{m} = m_x \hat{r} + m_\theta \hat{\theta} + m_\phi \hat{\phi} \) is a constant vector, and \( \vec{r} \neq 0 \).

A0.5 Check the theorem of gradients using \( T = x^2 + 4xy + 2yz^3 \), the points \( \vec{a} = (0,0,0) \), \( \vec{b} = (1,1,1) \), and the three paths in Fig. 1:

(a) \((0,0,0) \rightarrow (1,0,0) \rightarrow (1,1,1)\);

(b) \((0,0,0) \rightarrow (0,0,1) \rightarrow (1,1,1)\);

(c) (the parabolic path \( z = x^2 \); \( y = x \).)

[Figures 1(a), 1(b), 1(c)]

A0.6 Check the gradient theorem for function \( T = r (\cos \theta + \sin \theta \cos \phi) \) using the path shown in Fig. 2.

[Figures 2 and 3]
A0.7 Check the divergence theorem for the function \( \vec{v} = r^2 \cos \theta \hat{r} + r^2 \cos \varphi \hat{\theta} - r^2 \cos \theta \sin \varphi \hat{\phi} \), using as your volume one octant of the sphere of radius \( R \) (Fig. 3).

A0.8 Check the Stokes’ theorem using the function \( \vec{v} = (ay)\hat{x} + (bx)\hat{y} \) (\( a \) and \( b \) are constants) and the circular path of radius \( R \), centered at the origin in the \( xy \) plane.

A0.9 Check Stokes’ theorem for the function \( \vec{v} = y\hat{z} \), using the triangular surface shown in Fig. 4.

A0.10 Check the divergence theorem for the function \( \vec{v} = r^2 \sin \theta \hat{r} + 4r^2 \cos \theta \hat{\theta} + r^2 \tan \theta \hat{\phi} \), using the volume of the “ice-cream cone” shown in Fig. 5 (the top surface is spherical, with radius \( R \) and centered at the origin).

A0.11 Prove the following integral formulas:

(a) \( \int_S \nabla \cdot \vec{v} \, d\tau = \int_S \vec{v} \cdot d\vec{a} \);

(b) \( \int_S \nabla \times \vec{B} \, d\tau = \int_S \vec{n} \times \vec{B} \, d\vec{a} \);

(c) \( \int_S \vec{n} \times \nabla \psi \, d\vec{a} = \oint_{\partial S} \vec{r} \times \psi \, d\vec{l} \);

(d) \( \int_S (\vec{n} \times \nabla) \cdot \vec{v} \, d\vec{a} = \oint_{\partial S} (\vec{r} \times \vec{v}) \cdot \vec{n} \, d\vec{l} \);

(e) Green’s first identity: \( \int_V (\phi \nabla^2 \psi + \nabla \cdot \nabla \psi) \, d\tau = \int_S \phi \vec{n} \cdot \nabla \psi \, d\vec{a} \);

(f) Green’s theorem: \( \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, d\tau = \oint_{S} (\phi \nabla \psi - \psi \nabla \phi) \cdot \vec{n} \, d\vec{l} \);

\[ = \int_S (\frac{\partial \psi}{\partial n} - \frac{\partial \phi}{\partial n}) \, d\vec{a} \]

(g) \( \int (f \vec{f} \cdot \nabla \psi + g \vec{J} \cdot \nabla \phi + f g \nabla \cdot \vec{J}) \, d\tau = 0 \)

Where \( \vec{J}(\vec{r}') \) is localized but not necessarily divergenceless, \( f(\vec{r}') \) and \( g(\vec{r}') \) are well-behaved functions of \( \vec{r}' \).

(h) The integral \( \vec{a} \equiv \int_S d\vec{a} \) is sometimes called the vector area of the surface \( S \). If \( S \) happens to be flat, then \( |\vec{a}| \) is the ordinary (scalar) area, obviously.

- Find the vector area of a hemispherical bowl of radius \( R \).
- Show that \( \vec{a} = 0 \) for any closed surface. [Hint: use integral in A0.11(a)].
• Show that $\tilde{a}$ is the same for all surface sharing the same boundary.

• Show that $\tilde{a} = \frac{1}{2} \oint \vec{r} \times d\ell$, where the integral is around the boundary line.

[Hint: One way to do it is to draw the cone subtended by the loop at the origin. Divide the conical surface up into infinitesimal triangular wedges, each with vertex at the origin and opposite side $d\ell$, and exploit the geometrical interception of the cross product.]

(i) Show that $\oint_s (\vec{c} \cdot \vec{r}) d\ell = \tilde{a} \times \vec{c}$, where, $\tilde{a} = \int_s d\tilde{a}$ is the vector area of the loop surface, $\vec{r}$ is the position vector of a point on the loop with respect to some origin, $\vec{c}$ is any constant vector.

[Hint: Use integral $\int \hat{n} \times \nabla \psi da = \oint \psi d\vec{l}$ in A0.11(c), and product rule of $\nabla(\vec{A} \cdot \vec{B})$.]

A0.12
(a) Show that $\delta(kx) = \frac{1}{|k|} \delta(x)$;
(b) Show that $\delta(-x) = \delta(x)$;
(c) Show that $\delta'(-x) = -\delta'(x)$;
(d) Show that $x\delta'(x) = -\delta(x)$;
(e) Show that $\delta'(t - r / c) = -c^2 \delta'(r - ct)$;
(f) Show that $\delta^*(t - r / c) = c^3 \delta^*(r - ct)$;
(g) Show that $\int_{-\infty}^{\infty} f(x) \delta'(x - a) dx = -f'(a)$.

(h) Let $\theta(x)$ be the step function: $\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$, show that $\frac{d\theta}{dx} = \delta(x)$.

A0.13 Evaluate the following integrals:
(a) $\int_{x=0}^{1} (3x^2 - 2x - 1) \delta(x - 3) dx$
(b) $\int_{x=0}^{1} \cos x \delta(x - \pi) dx$
(c) $\int_{x=-1}^{1} x^3 \delta(x + 1) dx$
(d) $\int_{x=-1}^{1} \ln(x + 3) \delta(x + 2) dx$
(e) $\int_{x=0}^{1} (2x + 3) \delta(3x) dx$
(f) $\int_{x=0}^{1} (x^3 + 3x + 2) \delta(1 - x) dx$
(g) $\int_{x=0}^{1} 9x^2 \delta(3x + 1) dx$
(h) $\int_{x=-\infty}^{\infty} \delta(x - b) dx$

A0.14 Evaluate the following integrals:
(a) $\int_{all\ space} (r^2 + \vec{r} \cdot \vec{a} + a^2) \delta(\vec{r} - \vec{a}) d\tau$, where $\vec{a}$ is a fixed vector and $a$ is its magnitude.
(b) $\int_{V} |\vec{r} - \vec{b}|^2 \delta^3(5\vec{r}) d\tau$, where $V$ is a cube of side 2, centered on the origin, and $\vec{b} = 4\hat{y} + 3\hat{z}$.
(c) $\int_{V} (r^4 + r^2 (\vec{r} \cdot \vec{c}) + c^4) \delta(\vec{r} - \vec{c}) d\tau$, where $V$ is a sphere of radius 6 about the origin, $\vec{c} = 5\hat{x} + 3\hat{y} + 2\hat{z}$, and $c$ is its magnitude.
(d) \( \int_V \mathbf{r} \cdot (\mathbf{a} - \mathbf{r}) \delta^3 (\mathbf{e} - \mathbf{r}) d\tau \), where \( \mathbf{a} = (1,2,3) \), \( \mathbf{e} = (3,2,1) \), and \( V \) is a sphere of radius 1.5 centered at (2,2,2).

A0.15 Write down the expressions of \( \delta^3 (\mathbf{r} - \mathbf{r}') \) in Cartesian, Spherical and cylindrical coordinates.

A0.16 Using Dirac delta functions and theta functions in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities \( \rho(\mathbf{r}) \):

(a) In spherical coordinates, a charge \( Q \) is uniformly distributed over a spherical shell of radius \( R \).

(b) In cylindrical coordinates, a charge \( \lambda \) per unit length is uniformly distributed over a cylindrical surface of radius \( b \).

(c) In cylindrical coordinates, a charge \( Q \) spreads uniformly over a flat circular disc of radius \( R \) and negligible thickness.

(d) The same as part (c), but using spherical coordinates.

(e) In spherical coordinates, a charge \( Q \) is uniformly distributed over a circular ring of radius \( a \).

(f) In spherical coordinates, a charge \( Q \) is uniformly distributed over a half of a circular ring with radius \( R \) as shown in Fig. 6.

(g) In cylindrical coordinates, a charge \( Q \) is uniformly distributed over a half of a circular ring with radius \( R \) as shown in Fig. 6.

(h) In cylindrical coordinates, a charge \( \lambda \) per unit length is uniformly distributed over an infinite long straight wire.

(i) In spherical coordinates, a charge \( Q \) is uniformly distributed over a straight wire of length \( 2b \).

(j) In Cartesian coordinates, a charge \( \lambda \) per unit length is uniformly distributed over a straight wire of length \( l \) which lies on the positive x-axis from \( x = -l \) to \( x = 0 \).

(k) In Cartesian coordinates, a charge \( \lambda \) per unit length is uniformly distributed over a straight wire of length \( l \) which lies on the x-axis from \( x = -l/2 \) to \( x = l/2 \).

(l) In spherical coordinates, point charges are located as shown in the Fig. 7 (a) and (b).

(m) In spherical coordinates, a charge \( Q \) is uniformly distributed over a volume of the “ice-cream cone” shown in Fig. 5.

(n) In spherical coordinates, a charge \( Q \) is uniformly distributed over the spherical surface of the “ice-cream cone” shown in Fig. 5.

(o) In spherical coordinates, a charge \( Q \) is uniformly distributed over 1/8 of a spherical shell as shown in Fig. 3.
(p) In spherical coordinates, a charge \( Q \) is uniformly distributed over the volume of 1/8 of a sphere as shown in Fig. 3.

(q) In spherical coordinates, a line charge of length 2\( d \) with a total charge \( Q \) has a linear charge density varying as \(( d^2 - z^2 )\), where \( z \) is the distance from the middle point.

(r) In spherical coordinates, a charge density \( \sigma(\theta) = \beta \sin^2(\theta/2) \) \( ( \beta \text{ is a constant) } \)
is glued over the surface of a spherical shell of radius \( R \).

A0.17 Express the volume current density of a current loop of radius \( R \), lying in the \( x \)-\( y \) plane, centered at the origin and carrying a current \( I \), using Dirac delta functions in (a) spherical coordinates, and (b) cylindrical coordinates.

**Chapter 1**

Jackson: 1.1, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 1.10, 1.12, 1.13, 1.15

A1.1 Two spheres, each of radius \( R \) and carrying uniform charge densities \( + \rho \) and \( - \rho \), respectively, are placed so that they partially overlap (Fig. 1). Call the vector from the positive center to the negative center \( \vec{d} \). Show that the field in the region of overlap is constant, and find its value.

A1.2 Find the \( \vec{E} \)-field and potential \( \Phi \) inside and outside a uniformly charged solid sphere whose radius is \( R \) and whose total charge is \( q \). Use infinity as the reference point of \( \Phi \). Compute the gradient of \( \Phi \) in each region, and check that it yields the correct \( \vec{E} \)-field.

A1.3 A long coaxial cable (Fig. 2) carries a uniform volume charge density \( \rho \) on the inner cylinder (radius \( a \)), and a uniform surface charge density on the outer cylindrical shell (radius \( b \)). This surface charge is negative and of the right magnitude so that the cable as a whole is electrically neutral.

(a) Find the electric field in each of the three regions: (i) \( s < a \), (ii) \( a < s < b \), (iii) \( s > b \).

(b) Find the potential difference between a point on the axis and a point on the outer cylinder.

A1.4 A conical surface (an empty ice-cream cone) carries a uniform surface charge \( \sigma \). The height of the cone is \( a \), as is the radius of the top. Find the potential difference between point \( P \) (the vetex) and \( Q \) (the center of the top).

A1.5 A metal sphere of radius \( R \), carrying charge \( q \), is surrounded by a thick concentric metal shell (internal radius \( a \), outer radius \( b \), as in Fig. 3). The shell carries no net charge.

(a) Find the surface charge density \( \sigma \) at \( R \), at \( a \), and at \( b \).

(b) Find the potential at the center, using infinity as the reference point.
(c) Now the outer surface is touched to a grounded wire, which lowers its potential to zero (same as at infinity). How do your answers to (a) and (b) change?

A1.6 An inverted hemispherical bowl of radius $R$ carries a uniform surface charge density $\sigma$. Find the potential difference between the “north pole” and the center.

A1.7 A sphere of radius $R$ carries a charge density $\rho(r) = kr$ (where $k$ is a constant).

(a) Find the $E$-field at: $r < R$ and $r > R$ respectively.

(b) Find the energy of the configuration using two different ways:

$$U_{E} = \frac{\varepsilon_{0}}{2} \int_{all\space space} E^{2} d\tau$$

and

$$U_{E} = \frac{1}{2} \int \rho(r') \Phi(r') d\tau'$$

A1.8 Two infinitely long wires running parallel to the $x$ axis carry uniform charge densities $+\lambda$ and $-\lambda$ respectively and are separated by a distance $R$ as shown in Fig. 4.

(a) Find the potential at any point $(x,y,z)$, using the origin as your reference.

(b) Show that the equipotential surfaces are circular cylinders, and locate the axis and radius of the cylinder corresponding to a given potential $V_{0}$.

A 1.9 Derive the following formulas

(a) Use Gauss’s law (draw a diagram) to derive $E_{2} - E_{1} = \frac{\sigma}{\varepsilon_{0}} \hat{n}$ at a boundary (Fig. 5)

(b) Use Faraday’s law (draw a diagram) to derive $\hat{n} \times (E_{2} - E_{1}) = 0$ at a boundary.

(c) Force on a surface charge $\sigma$ of a conductor: $f = \frac{\sigma^{2}}{2\varepsilon_{0}} = \hat{n} \frac{\varepsilon_{0}}{2} E^{2} = \hat{n} u_{e}$
Chapter 2

Jackson: 2.1, 2.2, 2.3, 2.4, 2.5, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.23

A2.1 A uniform line charge $\lambda$ is placed on an infinite straight wire, a distance $d$ above a grounded conducting plane. (Let’s say the wire runs parallel to the x-axis and directly above it, and the conducting plane is the xy plane).

(a) Find the potential in the region above the plane.

(b) Find the charge density $\sigma$ induced by the conducting plane.

A2.2 A conducting sphere of radius $a$ is placed in a uniform electric field $\vec{E}_0 = E_0 \hat{z}$. The uniform $\vec{E}$-field can be thought of as being produced by $\pm Q$ at $-R$, then in the region near the origin whose dimensions are very small compared to $R$ there is an approximately constant filed $E_0 = \frac{2kQ}{R^2} = \frac{Q}{2\pi\varepsilon_0 R^2}$ parallel to the z-axis. In the limit as $R, Q \to \infty$ with $\frac{Q}{R^2} = \text{constant}$, this approximation is exact.

(a) Use the method of image to find the electric potential $\Phi(\vec{r})$ outside the sphere, and show that the two leading terms in $\Phi(\vec{r})$ are due to $\vec{E}_0$ and electric dipole moment induced on the conducting sphere respectively.

(b) Check that $\Phi(\vec{r})$ satisfies $\nabla^2 \Phi(\vec{r}) = -\rho / \varepsilon_0$.

A 2.3 Two long straight wires, carrying opposite uniform charges $\pm \lambda$, are situated on either side of a long conducting cylinder (Fig. 1). The cylinder (which carries no net charge) has radius $b$, and the wires are a distance $R$ from the axis. Find potential at point $P$.

A2.4 Two infinitely long grounded metal plates, at $y=0$ and $y=a$, are connected at $x=\pm b$ by metal strips maintained at a constant potential $V_0$ as shown in Fig. 2 (a thin layer of insulation at each corner prevents them from shorting out). Find the potential inside the rectangular pipe.
A2.5 An infinitely long rectangular metal pipe (side a and b) is grounded, but one end, at x=0, is maintained at a special potential $V_0(y,z)$, as indicated in Fig. 3. Find the potential inside the pipe.

A2.6 Two infinite grounded metal plates lie parallel to the xz plane, one at y=0, the other at y=a (Fig. 4). The left end, at x=0, consists of two metal strips: one, from y=0 to y=a/2, is held at a constant potential $V_0$, and the other, from y=a/2 to a, is at potential $-V_0$. Find the potential inside the infinite slot.

A2.7 Charge density $\sigma(\varphi) = a \sin 5\varphi$ (where a is a constant) is glued over the surface of an infinite cylinder of radius R (Fig. 5). Find the potential inside and outside the cylinder.

A2.8 A long cylindrical shell of radius R carries a uniform surface charge density $\sigma_0$ on the upper half and an opposite charge $-\sigma_0$ on the lower half (Fig. 6). Find the electric potential inside and outside the cylinder.

A2.9 Find the potential outside an infinitely long metal pipe, of radius R, placed at right angles to an otherwise uniform electric field $\vec{E}_0$. Find the surface charge induced on the pipe.

A2.10 Use the Green’s function $G_d(s, s') = \ln \frac{s^2 + b^4 - 2s's\cos(\varphi - \varphi')}{b^2 \left[ s^2 + s'^2 - 2ss'\cos(\varphi - \varphi') \right]}$ to find the potential inside a cylinder of radius b with $\Phi(b, \varphi')$ specified on the cylinder.

A2.11 Use separation of variables to find the potential everywhere outside an infinitely long cylinder of radius R with the following boundary conditions:

- $\Phi(R, \varphi) = +V_0$ for $0 < \varphi < \pi$, and
- $\Phi(R, \varphi) = -V_0$ for $\pi < \varphi < 2\pi$.

There are no charges inside and outside the cylinder.
A 2.12 Find the potential in the interior of a sphere where \( \Phi = V_0 \) on the surface of the sphere for \( 0 \leq \theta \leq \pi/2, \), \( 0 \leq \phi \leq \pi/4 \) and \( \Phi = 0 \) on the rest of the spherical surface of radius \( R \). (as shown in the figure). The result may be left in the form of an integral involving no vector symbols and with all limits of integration given clearly.

A 2.13 Use the Green’s function for \( \nabla^2 \) operator with spherical boundary \( G(r,r') = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \gamma}} - \frac{1}{\sqrt{R^2 + r'^2 - 2rr' \cos \gamma}} \) to find the potential outside a conducting sphere of radius \( R \) made up of two hemispherical shells, separated by a small insulating ring which lies in the \( z=0 \) plane, with the upper (lower) hemisphere at potential \( +V \) (\( -V \)). (You don’t need to carry out the integral in the final expression of \( \Phi(r,\theta,\phi) \)).

A 2.14 Using the method of images, discuss the problem of a point charge \( q \) outside a hollow, grounded, conducting sphere of inner radius \( a \). The charge is located at \( d \) (\( d > a \)) from the origin as shown in the figure.

(a) Find the electric potential \( \Phi(r,\theta,\phi) \) at point \( P \) outside the sphere. (You need to derive the formulas for the image charge \( q_i \) and its position \( b \) in terms of \( a, d, q \)).

(b) Verify that \( \Phi(r,\theta,\phi) \) satisfies the Poisson equation (show your work).

(c) Calculate the force on the point charge \( q \).

(d) Calculate the charge density induced on the outer surface of sphere.

(e) Calculate the total charge induced on the outer surface of sphere.
A point charge $q$ is located at a distance $d$ from the center of a conducting sphere with radius $R$ ($R < d$). The conducting sphere is isolated and charged with $Q$ as shown in the figure.

(a) Find the electric potential $\Phi(r, \theta, \phi)$ at point $P$ outside the sphere.

(b) Verify that $\Phi(r, \theta, \phi)$ satisfies the Poisson equation (show your work).

(c) Calculate the force on $q$. 

\[ Q \]

\[ P(r, \theta, \phi) \]

\[ d \]

\[ q \]
Chapter 3

Jackson: 3.1, 3.2, 3.3, 3.4, 3.6, 3.7, 3.9, 3.10, 3.12, 3.14, 3.17, 3.22, 3.24

A3.1 The potential \( V_0(\theta) = C \sin^2(\theta/2) \) is specified on the surface of a hollow sphere, of radius \( R \). Find the potential inside and outside the sphere.

A3.2 An uncharged metal sphere of radius \( R \) is placed in an otherwise uniform electric field \( \vec{E} = E_0 \hat{z} \). The field will push positive charge to the “northern” surface of the sphere, leaving a negative charge on the “southern” surface (as shown Fig. 1). This induced charge, in turn, distorts the field, in the neighborhood of the sphere.

(a) Use separation of variables to find the potential in the region outside the sphere.

(b) Find the charge density on the surface of the sphere.

A3.3 The potential \( V_0 = \beta (\frac{1}{2} \cos \theta + \cos^2 \frac{\theta}{2}) \) is specified on the surface of a hollow, empty sphere of radius \( R \). Find the potential inside the sphere. (\( \beta \) is a constant).

A3.4 Use separation of variables to find the potential (a) inside and (b) outside a conducting sphere of radius \( R \) made up of two hemispherical shells, separated by a small insulating ring which lies in the \( z=0 \) plane, with the upper (lower) hemisphere at potential \(+V\) (\(-V\)), (Fig. 2).

A3.5 A specified charge density \( \sigma_0(\theta) = C \cos \theta \) is glued over the surface of a spherical shell of radius \( R \). Find the resulting potential inside and outside the sphere.

A3.6 Use \( \Phi(r, \theta) = \Phi(r = z) P_z (\cos \theta) \) to find the potential outside a conducting sphere outside a conducting sphere of radius \( R \) made up of two hemispherical shells, separated by a small insulating ring which lies in the \( z=0 \) plane, with the upper (lower) hemisphere at potential \(+V\) (\(-V\)), (Fig. 2).

A3.7 Find the potential due to a total charge \( q \) uniformly distributed around a circular ring of radius \( a \), located as shown in the figure, with its axis the \( z \)-axis and its center at \( z=b \), (Fig. 3).
A3.8 Find potential inside the cylinder shown in the Fig. 4.

A3.9 Use separation of variables to find the potential between the two concentric cylindrical boxes as shown in Fig. 5. The radii of the inner and outer cylinders are a and b respectively. The boundary conditions are:

\[ \Phi = 0 \quad \text{at } s=a; \]
\[ \Phi = V(b,\phi,z) \quad \text{at } s=b; \]
\[ \Phi = 0 \quad \text{at } z=0; \]
\[ \Phi = 0 \quad \text{at } z=L; \]

A3.10 Find potential (a) inside, (b) outside the cylinder and between the two parallel zero potential planes as shown in the Fig.6.

A3.11 A hollow grounded sphere with a uniform line charge of total Q located on the z-axis between the north and south poles of the sphere. Find the potential inside the sphere. (Shown in Fig.7).

A3.12 Find the potential inside a hollow grounded sphere of radius b with a concentric ring of radius a. The circular ring is uniformly charged with a total charge Q and located in the x-y plane as shown in the Fig.8.

A3.13 Give the Green function for \( \nabla^2 \), for the interior of a hemisphere and homogeneous Dirichlet BC as a sum over spherical harmonics. (Fig. 9).

A3.14 Give the Green function for \( \nabla^2 \) in the interior of a rectangular region of side \( a \times b \times c \) with homogeneous Dirichlet BC. Give the Green function in terms of a sum over eigenfunctions with all parameters and sums clearly labeled. (Fig. 10)
Chapter 4

Jackson: 4.1, 4.2, 4.5, 4.6, 4.7(a)(b), 4.8, 4.9, 4.10, 4.13

A4.1 A charge \( +Q \) is distributed uniformly along the \( z \)-axis from \( z = -a \) to \( z = +a \). Show that the electric potential at a point \( \mathbf{r} \) is given by

\[
\Phi(r, \theta) = \frac{q}{4\pi \varepsilon_0 r} \left[ \frac{1}{3} \left( \frac{a}{r} \right)^2 P_2(\cos \theta) + \frac{1}{5} \left( \frac{a}{r} \right)^4 P_4(\cos \theta) + \ldots \right], \quad \text{for } r > a.
\]

A4.2 Consider a dielectric sphere of radius \( R \), (linear homogeneous dielectric \( \varepsilon \)), in which there is a uniform free charge density \( \rho_f \) out to a radius \( a < R \).

(a) Compute \( \mathbf{D} \) everywhere in space.

(b) Compute \( \mathbf{E} \) and \( \mathbf{P} \) everywhere.

(c) Find \( \sigma_{\text{pol}} \) and \( \rho_{\text{pol}} \).

(d) Show that the electric field \( \mathbf{E} \) can be regarded as arising from all charges present with no reference to any matter.

A4.3 A sphere of radius \( R \), centered at the origin, carries charge density \( \rho(r, \theta) = k \frac{R}{r^2} (R - 2r) \sin \theta \), where \( k \) is a constant, and \( r, \theta \) are the usual spherical coordinates. Find the approximate potential for points on the \( z \)-axis, far from the sphere.

A4.4 A charge density \( \sigma = \beta \cos \theta \) (\( \beta \) is a constant) is glued over the surface of a spherical shell of radius \( R \). Calculate

(a) all the multipole moments \( q_{lm} \) of the charge distribution.

(b) the potential outside the sphere due to all the \( q_{lm} \).

A4.5 A (perfect) dipole \( \mathbf{p} \) is situated a distance \( z \) above an infinite ground conducting plane (Fig. 1). The dipole makes an angle \( \theta \) with the perpendicular to the plane. Find the torque on \( \mathbf{p} \). If the dipole is free to rotate, in what orientation will it come to rest?

A4.6 A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field \( \mathbf{E}_0 \). Find the electric field inside the sphere.

A4.7 Suppose the entire region below the plane \( z = 0 \) in Fig. 2 is filled with uniform linear dielectric material of susceptibility \( \chi \). Calculate the force on a point charge \( q \) situated a distance \( d \) above the origin.
A4.8 A very long cylinder (with radius R) of linear dielectric material (with susceptibility $\chi_e$) is placed in an otherwise uniform electric field $E_0$, which is perpendicular to the axis of the cylinder. Find the resulting field within the cylinder.

A4.9 An uncharged conducting sphere of radius a is coated with a thick insulating shell (dielectric constant $\varepsilon_r$) out to radius b. This object is now placed in an otherwise uniform electric field $E_0$. Find the electric field in the insulator.

A4.10 A certain coaxial cable consists of a copper wire, radius a, surrounded by a concentric copper tube of internal radius c (Fig. 3). The space between is partially filled (from b to c) with material of dielectric constant $\varepsilon_r$, as shown. Find the capacitance per unit length of the cable.

A4.11 In Fig. 4, $\vec{p}_1$ and $\vec{p}_2$ are (perfect) dipoles a distance r apart.

(a) What is the force on $\vec{p}_2$ due to $\vec{p}_1$?
(b) What is the torque on $\vec{p}_2$ due to $\vec{p}_1$ (with respect to the center of $\vec{p}_1$)?
(c) What is the force on $\vec{p}_1$ due to $\vec{p}_2$?
(d) What is the torque on $\vec{p}_1$ due to $\vec{p}_2$ (with respect to the center of $\vec{p}_1$)?
(e) Are the answers in (a) and (b), (c) and (d) consistent with Newton’s third law?

A4.12 Find the electric fields at $r < R$ and $r > R$ produced by a uniformly polarized sphere of radius R with $\vec{P} = C\hat{z}$, where C is a constant.

A4.13 A spherical cavity of radius R is in an infinitely large dielectric medium with dielectric constant $\varepsilon_r = \varepsilon / \varepsilon_0$ and with an applied field $E_0$ parallel to the z-axis. Find the electric fields inside and outside the cavity.

A4.14 Suppose you have enough linear dielectric material, of dielectric constant $\varepsilon_r$, to half-fill a parallel-plate capacitor (Fig. 5) (The area of each plate is A, the separation between the two plates is d). If the free charges ($\pm Q$) on the two plates are fixed, find $\vec{D}$, $\vec{E}$, $\vec{P}$, in each region, and $\sigma_f$ and $\sigma_{pol}$ on the surfaces related to the plate with $+Q$. By what fraction is the capacitance increased when the dielectrics is inserted?
A4.15 A point charge \( q \) is located at a distant \( d \) away from an infinite plane interface between two linear homogeneous dielectric materials with electric permittivity \( \varepsilon_1 \) and \( \varepsilon_2 \) respectively as shown in Fig. 6.

(a) Write down \( \rho_f \) and \( \rho_{pol} \) in each dielectrics.

(b) Write down the Poisson equation in each dielectrics.

(c) Write down the boundary conditions.

(d) Use method of image to find the electric potential in each dielectrics.

(e) Check that the solutions found in (b) satisfied the Poisson equations found in (a).

(f) Find \( \sigma_{pol} \) on the interface.

\[ \varepsilon_2 \quad \varepsilon_1 \quad q \]

\[ d \]

Fig. 6