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Correlated walks down the Babylonian markets

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Abstract – To investigate the evolution of market dynamics in different stages of historical development, we analyze commodity prices from two distinct periods —ancient Babylon, and medieval and early modern England. We find that the first-digit distributions of both Babylon and England commodity prices follow Benford's law, indicating that the data represent empirical observations typically arising from a free market. Further, we find that the normalized prices of both Babylon and England agricultural commodities are characterized by stretched exponential distributions, and exhibit persistent correlations of a power law type over long periods of up to several centuries, in contrast to contemporary markets. Our findings suggest that similar market interactions may underlie the dynamics of ancient agricultural commodity prices, and that these interactions may remain stable across centuries in two distinct historical periods.

The emergence of markets and market economics is an active area of research in archeology and economic anthropology, where a main focus is to understand how markets developed in early civilizations and what their characteristics were [1–4]. A plausible hypothesis is that markets emerge in certain societies as a result of the exchange of goods, services or information, allowing a particular distribution of resources. Buyers and sellers, as primary market participants, exert demand and supply forces responsible, among others, for driving the price of any asset. The complex interactions between numerous market agents acting through feedback at different time scales within various economic conditions and market regulation lead to highly irregular and complex dynamics of market activity [5–10]. Recent empirical investigations have demonstrated that key market observables such as price, trading volume and frequency of trading do not change in a random manner but rather exhibit surprisingly robust dynamical patterns over a wide range of time scales described by scaling laws [11–13]. To understand how these scaling laws relate to the underlying market regulatory mechanisms and interactions among market entities, most studies have focused on high-frequency recordings of modern market activity as represented by commodities, company stocks or currency foreign exchange over relatively short time periods ranging from months and years up to several decades [14–16]. However, every market is embedded in its historical, cultural and technological context, and market dynamics may evolve with changes in economic conditions, government politics and market regulations —*e.g.*, correlations in stock price fluctuations change significantly when a company is transferred from one stock market to another [17], while networks of interaction of company stocks across the entire economy exhibit stable behavior over a limited time horizon [18,19]. The evolution of market dynamics across different historical periods has not been systematically studied. Here we investigate several key aspects of the dynamics of commodity prices in two distinct historical periods corresponding to different economic conditions and development of society —ancient Mesopotamia and medieval and early modern England.

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We study the probability distribution and correlation properties of commodity prices from Babylon in the period of 463–72 B.C. and from England in the period of 1209–1914 A.D. These data sets represent an exceptionally long time window into the market dynamics at two important and different periods of civilization. By comparing, in an statistical framework, key measures of the market dynamics during these two periods, we can identify important similarities or dissimilarities of their respective economy in relation to empirical observations of contemporary markets.

Situated in the south region of Iraq, on the Euphrates River, Babylon was the political and cultural capital of ancient Mesopotamia. Archaeological excavations started in 1899 by Robert Koldewey have uncovered astronomical diaries, written in cuneiform on clay tablets, from the period 652 B.C. to 69 A.D. Since the tablets contain celestial observations, all information inscribed on them is dated, including records of the weather, the level of the Euphrates River, socio-political events and market quotations of six commodities: barley, dates, mustard (cascuta), cress (cardamom), sesame and wool. The commodity prices are expressed in weight quantities that could be purchased per shekel of silver, recorded three times a month. Babylon had an agriculturally based economy and its diet was primarily based on barley and dates. Mustard and cress were used as spices and sesame primarily for its oil. All six commodities were also used in official rituals of Babylonian cultic life [20]. Many clay tablets are still missing or broken. Since they were stored vertically and the prices were listed last in the monthly reports, *i.e.*, at the lower edge of each clay tablet, there is a considerable amount of data lost.

The English commodity annual price records constitute a point of reference in many historical analyses, because they represent a data set over a significantly long period. In our study, we use records from 80 agricultural and other commodities traded in England in the period 1209–1914 [21]. Prices are expressed in grams of silver per metric physical unit. To compare with Babylon data, we select six similar agricultural commodities: barley, beans, oats, peas, pepper and wheat, for which there are practically continuous annual records throughout the entire period (with relatively few missing data points mainly in the first half of the period).

Most archaeologists, epigraphers, and economic historians have inferred that the Babylonian data represent real and accurate price quotations [2,4,22]. However, there is no conclusive evidence and it is possible that the Babylon tablets record price estimates, price caps, target prices, or even astrological calculations of what prices ought to be.

Numerical records of empirical observations arising from natural or social processes often exhibit a particular probability distribution for the first significant digit of the form $\log_{10}(1+1/n)$, where $n=1,\ldots,9$ is the first digit, known as Benford's law [23,24]. Benford's law has been observed in several financial systems like stock market prices [25], census statistics [26], income tax payments, and accounting data [27]. However, numerical processes influenced by human factors such as advertised prices for consumers, assigned telephone or license plate numbers, amount of cash withdrawals, or randomly picking numbers do not follow Benford's law [26–28]. Benford's test has been utilized in identifying fraudulent reports in taxes [27], transactions of federal campaigns [29], toxic release reported by industrial plants [30], accounting audits [31] and fabricated survey answers [32]. Therefore, a distribution of the first-digits which does not follow Benford's law could indicate the presence of systematic omissions, estimations, rounding, falsification or even fabrication in the recorded data. The first-digit distribution of Babylon commodity prices could thus expose certain abnormalities, which could lead one to suspect that these prices are not a record of real empirical observations typically arising from a free market where supply and demand forces are present.

Our analysis of the first significant digits of the Babylon commodity price records indicates a good agreement with Benford's law (fig. 1a) with the coefficient of determination $R^2=0.93$. These results are comparable with our analysis of six similar agricultural commodities from medieval and early modern England, which also conform to Benford's distribution (fig. 1b) (with $R^2=0.94$). Extending our analysis to 80 different English commodities of mixed (not only agricultural) nature to significantly increase the data samples, we obtain an almost perfect fit to Benford's law (fig. 1c) (with $R^2=0.95$). These findings indicate that Babylonian market quotations exhibit properties shared by empirical observations of natural process, suggesting they represent reliable recordings of commodity prices. Further, since multiplicative process have been associated with emergence of Benford's distribution [33] and since contemporary stock market prices also follow Benford's law [25], our results in fig. 1 may indicate that processes of multiplicative nature underlie the market dynamics in distinct historical periods and economic conditions.

Next, we study the probability distributions of different agricultural commodity prices in Babylon and we compare them with the prices of similar commodities from medieval and early modern England. We find that the price distribution of each Babylonian commodity i is characterized by different average $\langle S_i(t)\rangle$ and standard deviation σ_i. However, after normalization of the prices $S_i(t)$ to $\tilde{S}_i(t)=(S_i(t)-\langle S_i(t)\rangle/\sigma_i$, where $\langle S_i(t)\rangle$ is the average recorded price of that commodity, the distributions of all Babylonian commodities fall onto a single curve. This curve is well fit by a stretched exponential function:

$$P(\tilde{S})=\int_{\tilde{S}}^{\infty} p(x)\mathrm{d}x \;\sim\; e^{-(\frac{\tilde{S}}{\tau})^{\delta}}, \qquad (1)$$

where $p(x)$ is the probability density, $\delta=1.47$ is the stretch exponent and $\tau=1.60$ is a characteristic constant (with $R^2=0.97$). Interestingly, after the same price-normalization procedure, all cumulative distributions for

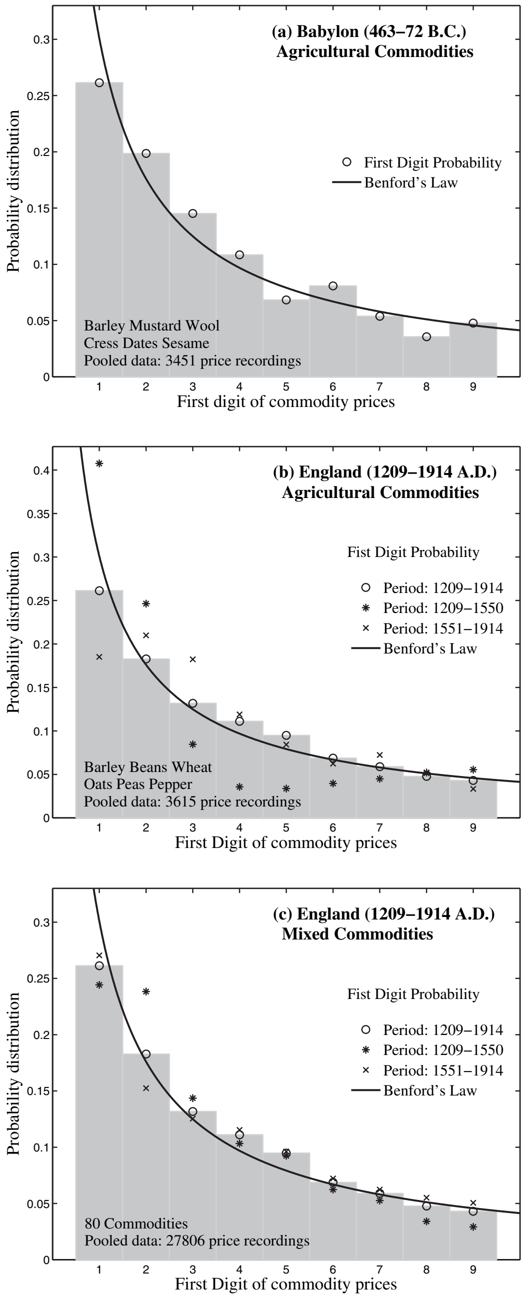

Fig. 1: First-digit distribution of the price records of (a) six agricultural commodities from Babylon, (b) six similar agricultural commodities from England and (c) 80 mixed commodities from England. All first-digit distributions are well approximated by Benford's law.

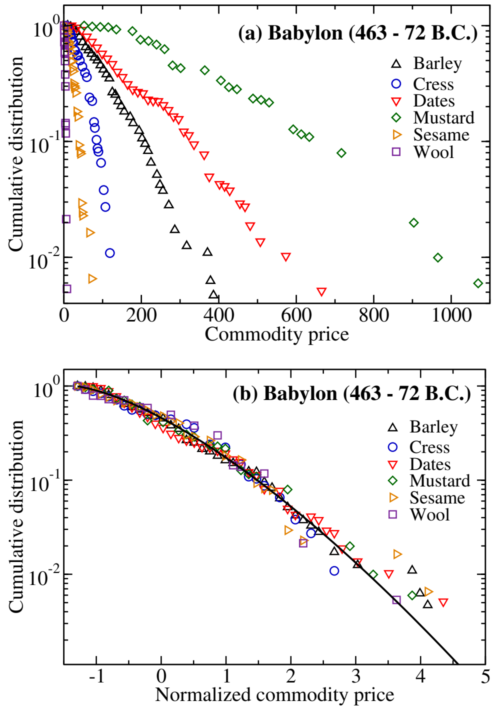

Fig. 2: (Colour on-line) Cumulative distributions of (a) agricultural commodity prices from ancient Babylon and (b) normalized prices of the same commodities as in (a). After normalization all distributions conform to a single stretched exponential curve, indicating a common functional form.

the six English commodities also fall onto a single curve following the same stretched exponential form in eq. (1) with parameters $\delta = 1.52$ and $\tau = 1.58$ (with $R^2 = 0.96$) very similar to those of the Babylonian prices. We find the same form for the probability distributions obtained for the first and the second half of the recorded period for the England data (see inset in fig. 3b), indicating time stability. Further, our analysis shows that contemporary agricultural commodity prices of wheat, barley, sugar, cocoa, coffee, tea (monthly prices for the period 1983–2009 [34]) as well as of gold (original monthly prices and prices after correction for inflation in the period 1971–2010 [35]) also exhibit a stretched exponential form for the cumulative distribution of normalized prices.

Our findings of identical functional form for the probability distribution for the agricultural commodity prices in ancient Babylon and medieval and early modern England, with very similar values for the parameters δ and τ, may suggest unexpected similarities underlying the dynamics for these two markets representing very different stages in historic and economic development. Our finding of stretched exponential distributions in fig. 2 and fig. 3 is in contrast to modern commodity and stock markets where probability distributions with power law tails have been reported [36–38], indicating a scale-invariant organization in commodity prices emerging from possibly different market organization and interactions of market participants. We note, that while most studies of contemporary markets focus on price returns, *i.e.*, the normalized forward change of the logarithm of the price at successive time's separated by a fixed time interval (*e.g.*, 1 min, 1 hour or 1 day), it is not possible to systematically define price returns for the Babylon commodities due to a large fraction of randomly lost data in the records [20,39].

Another key characteristic of market dynamics are the correlations and scaling behavior embedded in the

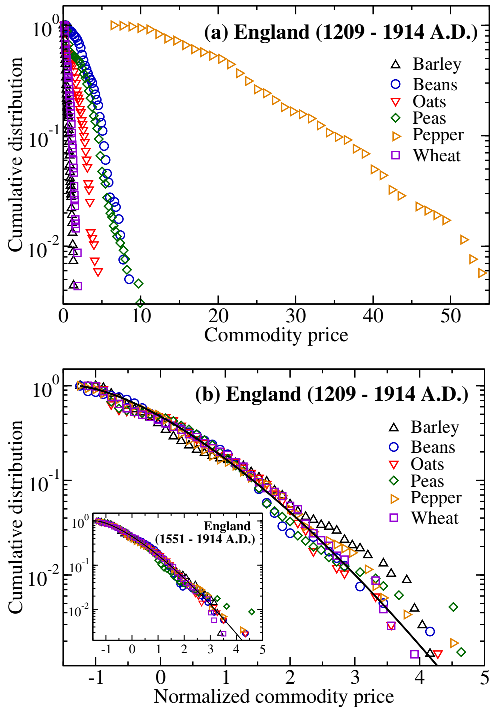

Fig. 3: (Colour on-line) Cumulative distributions of (a) agricultural commodity prices from medieval and early modern England and (b) normalized prices of the same commodities as in (a). All normalized distributions are well approximated by a stretched exponential function with parameter values similar to those for the Babylonian prices in fig. 2.

fluctuations of prices and other market observables. These characteristics reflect aspects of the temporal organization of multiple feedback interactions between many market agents, important to properly identify and model the underlying mechanisms of market regulation. To quantify the temporal structure in commodity prices in ancient Babylon and to compare it with the market dynamics of medieval and early modern England, we apply the detrended fluctuation analysis (DFA) [40]. Unlike other traditional methods such as power spectrum, autocorrelation and R/S analysis which are not well suited for nonstationary signals, the DFA method can accurately quantify correlations in the fluctuations of nonstationary signals generated by systems exhibiting nonequilibrium dynamics, with multiple degrees of freedom and nonlinear feedback interactions as observed in commodity and financial markets [6,7,9]. Moreover, recent studies of the performance of the DFA method have demonstrated that the method is robust to the presence of gaps and missing data in correlated signals [41,42].

The Babylon records contain commodity price data for the beginning, middle and end of each month. Since some of these records are missing, in our analysis we consider only monthly averaged and annually averaged prices. Applying the DFA, we calculate the root-mean-square fluctuation function $F(n)$ of the integrated and piece-wise polynomially detrended price time series for a given time window of size n (where n can be in units of months or years), and we obtain the functional dependence of $F(n)$ for varied time scale n. A power law dependence $F(n) \sim n^{\alpha}$ indicates the presence of scale-invariant (scaling) behavior, while the scaling exponent α (a self-similarity parameter) quantifies the long-term power law correlations in the data. If $\alpha = 0.5$, the signal is uncorrelated (white noise); if $\alpha < 0.5$, the signal is anti-correlated; if $\alpha > 0.5$, the signal is positively correlated. The larger the value of α, the stronger the correlations. In this study, we use the second-order DFA (DFA-2), which removes both constant and linear trends in the time series of commodity prices [43]. The choice of DFA-2 is motivated by the fact that: i) this order of DFA-l can accurately quantify the scaling behavior of signals with exponents in the range $0 < \alpha < 3$ [44], which covers practically all signals generated by real world systems; ii) earlier investigations have demonstrated that DFA-2 is sufficient to accurately quantify a broad range of nonstationary signals generated by different nonlinear dynamics —*e.g.*, for commodity and stock returns [37] and for intertrade times dynamics [45] the exponent α obtained from higher-order DFA-l is not significantly different compared to α obtained from DFA-2; and iii) deviations from scaling which appear at small scales become more pronounced in higher-order DFA-l [43].

We find that all six Babylon commodities exhibit power law correlations characterized by scaling exponent $\alpha \approx 0.7$ (fig. 4a). Moreover, the scaling exponent remains stable when changing the time scale from months to years when considering monthly and annually averaged commodity prices —$\alpha = 0.67 \pm 0.04$ (mean $\pm$ standard deviation of all six commodities) for the monthly data and $\alpha = 0.69 \pm 0.05$ for the annual data (see fig. 4a, filled and open symbols respectively). For the six agricultural English commodities (which we have selected to be similar in kind of the six Babylon commodities), we also find persistent power law correlation for their annual price fluctuations characterized by a scaling exponent $\alpha = 0.91 \pm 0.05$ over a broad range of few years to almost two hundred years (fig. 4b). Repeating our scaling analysis for the first and second half of the time period 1209–1914 A.D., we find a consistent positively correlated behavior, with an exponent $\alpha \approx 0.9$, for all six English commodities. Specifically, for the period 1209–1550 we find: barley $\alpha = 0.84$; oats $\alpha = 0.82$; peas $\alpha = 0.77$; pepper $\alpha = 0.93$; wheat $\alpha = 0.75$. For the period 1551-1914 we find: barley $\alpha = 0.92$; beans $\alpha = 0.85$; oats $\alpha = 0.95$; peas $\alpha = 0.91$; pepper $\alpha = 1.16$; wheat $\alpha = 0.82$.

Thus, our findings indicate that both ancient Babylonian and medieval English commodity prices exhibit the same kind of persistent power law correlations, suggesting common elements in the market mechanisms driving the price dynamics. Notably, our observations indicate a very different temporal organization of Babylon and medieval England price fluctuations compared to price fluctuations

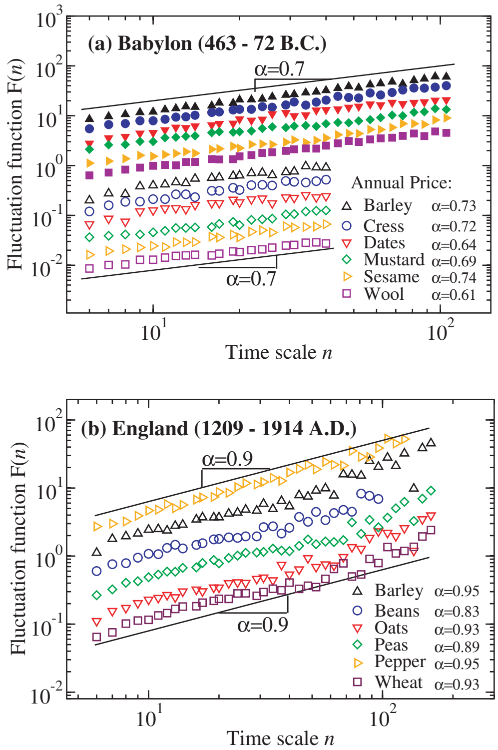

Fig. 4: (Colour on-line) Scaling analysis of (a) six agricultural commodity prices from ancient Babylon and (b) six similar agricultural commodity prices from medieval and early modern England. Open symbols in (a) represent annually averaged data and closed symbols represent monthly averaged data. Both Babylon and England data exhibit scaling behavior characterized by DFA scaling exponent $\alpha > 0.5$ indicating strong positive correlations.

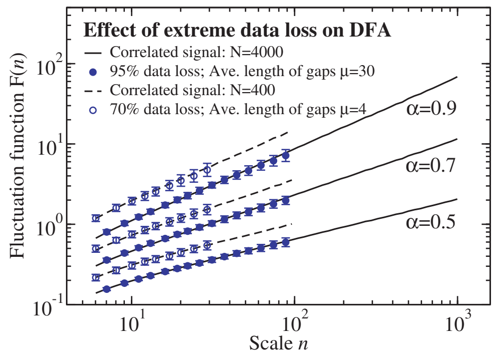

Fig. 5: (Colour on-line) Modeling the effect of extreme data loss on the scaling properties of correlated signals. Results from DFA-2 analysis for i) signals of length $N = 4000$ with different scaling exponents α and 95% of data loss where the average length of missing segments $\mu = 30$ data points (filled circles) and ii) signals of length $N = 400$ with different α and 70% of data loss where the average length of missing segments (gaps) is $\mu = 4$ data points (open circles). The missing segments in the simulations are drawn from an exponential distribution as observed in the Babylon data. Solid and dash lines indicate DFA-2 scaling before segments are removed, serving as a base line, and symbols indicate mean values of 100 different realizations with error bars showing the standard deviations. The simulations indicate that signals with long-term persistent correlations are not affected by a significant loss of data. The parameters in the simulations are chosen to represent the monthly (filled circles) and annually (open circles) averaged prices in the Babylon records.

of commodities traded on contemporary markets, where normalized price returns were found to be uncorrelated with $\alpha \approx 0.5$ for both spot and future commodity prices in contrast to the absolute price returns (volatility) where persistent long-range power law correlations were observed [37]. Indeed, our correlation analysis shows that contemporary agricultural commodity prices exhibit close to random walk behavior with $\alpha \approx 1.5$ corresponding to uncorrelated white noise behavior ($\alpha \approx 0.5$) for the price returns. Specifically, for the monthly prices in the period 1983–2009 [34] we find: wheat $\alpha = 1.38$; barley $\alpha = 1.37$; sugar $\alpha = 1.54$; cocoa $\alpha = 1.44$; coffee $\alpha = 1.50$; tea $\alpha = 1.54$. Further, for gold monthly prices in US dollars in the period 1971–2010 [35], we find $\alpha = 1.45$ for the original prices, and $\alpha = 1.54$ after correction for inflation. Our findings of strong persistent long-term correlations for the Babylon and England commodity prices (fig. 4) do not confirm earlier reports of random walk behavior for the commodities of these old markets [3].

We note, that due to significant gaps of missing data in the available Babylon commodity price records, it is not possible to perform a consistent correlation analysis for the price returns and absolute price returns. Because the missing data in the Babylon records constitute up to 70% of all annual prices and up to 95% of all monthly prices for the entire period for some of the commodities, we model the effect of extreme data loss on the scaling properties of correlated signals. We first determine the probability density function of the length of the missing gaps using the multihistogram method [46], and we find that the length of the gaps in both monthly and annually averaged data follows an exponential distribution, with mean $\mu \approx 30$ months and $\mu \approx 4$ years respectively. Considering the nature of the time series of the different Babylon commodity prices, we generate correlated signals with $\alpha > 0.5$, and we randomly remove a given percentage of the total data by cutting out segments drawn from an exponential distribution. To simulate the Babylon annual (average) prices, we generate correlated signals of length $N = 400$ data points, with DFA exponent $\alpha = 0.7$, and 70% missing data by randomly removing data segments with length drawn from an exponential distribution with mean $\mu = 4$. To simulate the monthly (average) prices of Babylon commodities, we generate correlated signals of length $N = 4000$ data points, with $\alpha = 0.7$ and 95% of missing data by randomly removing data segments from an exponential distribution with mean $\mu = 30$. Our simulations shown in fig. 5 demonstrate that even extreme loss of data does not significantly affect the scaling properties of positively correlated signals (for details on the effect of extreme data loss see [42]). These simulations validate our findings of long-term persistent correlations in both Babylon and England commodity prices.

In summary, our findings of stretched exponential form for the probability distributions and long-term persistent correlations of a power law type for the commodity prices in both ancient Babylon and medieval and early modern England, indicate strong similarity in the dynamics of these markets despite distinct differences in historical and economic conditions and development of society, wars and governmental disruptions in these two historical periods. Since these key statistical properties do not significantly change when we consider separate segments of the recordings, such consistency may suggest that the price dynamics of agricultural arable commodities are mainly driven by natural growth, weather conditions, population distribution and growth leading to similar supply and demand market forces and interactions which remain relatively stable across centuries in these two historical periods.

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